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ON ONE VARIATIONAL PRINCIPLE OF HAMILTON'S TYPE FOR NONLINEAR HEAT TRANSFER PROBLEM

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Abstract—In this paper a new Lagrangian for nonlinear heat conduction problem is constructed. Using the concept of penetration depth a computational procedure for solving the nonlinear heat equation is given. Problem with nonlinear boundary conditions (surface radiation) is also discussed. Applying the method of Yang [33–35], it is shown that the solutions can be improved. Also, the method of choosing the best trial polynomial for the description of the temperature distribution is discussed.

In the light of Yang's theory the solutions obtained by means of the variational principle have some degree of optimality in comparison to other approximative solutions.

I. INTRODUCTION

It is a well known fact, that the equations of heat transfer employed at the present time cannot be derived from exact Hamiltonian variational principles. However, there is always interest in describing the differential equations of the process under consideration as a variational problem. Contrary to the general impression that variational principles are of purely theoretical interest, the investigations in the past several years show, that they can lead to considerable practical advantages in the solution of various problems, especially nonlinear boundary value problems.

When a variational principle is established a general and systematic approximative procedure for establishing the solution can be developed from a direct study of the variational integral.

It is a well known fact [6] that when the Lagrangian of a physical problem exists it is possible to apply the very powerful technique of partial integration which is more effective and accurate than either the Ritz or Galerkin method. This fact is one of the principal justifications of applying the methods of variational principles in physics. Despite the fact that variational methods in dissipative physics have been criticised (see [1]), the increasing number of very successful applications of variational principles in obtaining approximate solutions, confirm the authors opinion that variational methods can be advantageously applied to transport phenomena (see for example [6, 10–13, 17]).

We will briefly mention the three leading variational principles in this area used in heat transfer analysis.*

(i) *Rosen's restricted variational principle*

The characteristic feature of this type of variational formulation is the presence of "frozen functions" which are held fixed during the process of variation. Immediately after the process of variation is finished the fixed functions become "unfrozen" i.e. equal to original functions of the problem. It should be noted that in some cases the Galerkin method will yield the same results as Rosen's principle. For more details and the chronological development of this idea the readers should consult the following references [2–7, 17].

* For variational principles concerning ideal fluid mechanics, as a related branch of continuum physics see [14].

(ii) *Bateman's hydrodynamical variational principle*

This principle was stated by Bateman [8] in the study of the flow of an incompressible viscous fluid, and elaborated by Morse and Feshbach [9] in the heat transfer problem. In this principle the concept of an image system is associated with the physical system. The image system has no physical significance. Consequently, there are no guides in picking boundary and initial conditions for the image system.

(iii) *Biot's variational principle* [10–13]

This principle does not have a Hamiltonian structure i.e. the Lagrangian of the problem does not exist. Using the concept of thermal potential, dissipation function, and generalized thermal forces a powerful tool was constructed for solving various equations describing nonlinear thermal phenomena.

This paper deals with a new variational principle for broad class of irreversible phenomena with applications to the nonlinear heat transfer problem. In order to be more pragmatic we will not only consider the formation of differential equations of the physical process, but also methods of obtaining approximative solutions using the direct method of the variational calculus. By means of the variational technique developed here a nonlinear boundary value problem can be reduced to an ordinary differential equation whose solution is often capable of being expressed in analytic closed form. The efficiency and accuracy of the method presented here can be estimated by comparing the solutions of problems solved using the variational method and other more elaborated methods. However, we shall study the problem of improving the accuracy of approximative solutions and choosing the best power of trial polynomial. Finally we will prove that the solution obtained with the help of the variational method discussed in this paper, has the property of optimality for a trial solution.

The examples treated are of considerable technical interest. Unfortunately, in many cases exact mathematical methods which apply to these problems are unique, difficult and frequently do not yield any results. However these difficulties can be avoided by using the variational method.

2. THE VARIATIONAL PRINCIPLE

Let the differential equations of a physical phenomenom be of the following form

$$\phi_i\left(x_j, t, v_i, \frac{\partial v_i}{\partial x_j}, \frac{\partial v_i}{\partial t}, \frac{\partial^2 v_i}{\partial x_j^2}\right) = 0 \tag{1}$$

$$i = 1, 2, \ldots, n; \quad j = 1, 2, 3$$

where v_i are a set of real field variables (velocity, pressure, temperature, etc.). The $v_i = v_i(x_j, t)$ are functions of the continuous cartesian variables x_j, and time t.

Let us consider the Lagrangian density [31]

$$\mathscr{L} = \mathscr{L}\left(x_j, t, v_i, \frac{\partial v_i}{\partial x_j}, \frac{\partial v_i}{\partial t}, \frac{\partial^2 v_i}{\partial x_j^2}, \lambda\right) \tag{2}$$

where λ is a parameter. For the Lagrangian density (2) we have the action integral

$$I = \int_{t_0}^{t_1} \int_V \mathscr{L} \, \mathrm{d}V \, \mathrm{d}t, \tag{3}$$

where $\mathrm{d}V = \prod_{j=1}^{3} \mathrm{d}x_j$ is an elementary volume in geometrical space. t_0, t_1 is an arbitrary interval of time.

If we restrict the variations in such a way that $\delta v_i|_V = 0$, on the boundaries of V for every time t, and $\delta v_i|_{t_0} = \delta v_i|_{t_1} = 0$ everywhere in V, including the boundaries of V, then the variational equation

$$\delta I = 0$$

is equivalent to the Euler–Lagrange equations for generalized coordinates v_i

$$\frac{\partial \mathscr{L}}{\partial v_i} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial(\partial v_i/\partial t)} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\frac{\partial \mathscr{L}}{\partial(\partial v_i/\partial x_j)}$$

$$+\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}\frac{\partial \mathscr{L}}{\partial(\partial^2 v_i/\partial x_j^2)}=0. \qquad (4)$$

$$i = 1, 2, 3, \ldots\ldots$$

For the Lagrangian (2), Euler–Lagrange equations (4) are:

$$\phi_i\left(x_j, t, v_i, \frac{\partial v_i}{\partial t}, \frac{\partial v_i}{\partial x_j}, \frac{\partial^2 v_i}{\partial x_j^2}\right) = \mathscr{N}_i\left(x_j, t, v_i, \frac{\partial v_i}{\partial x_j}, \frac{\partial v_i}{\partial t}, \frac{\partial^2 v_i}{\partial x_j^2}, \ldots, \lambda\right). \qquad (5)$$

Where the left hand side of (5) is the same as the left hand side of (1). In order to get correct differential equations (1) we will impose the condition that the parameter λ tends to zero. In this case the functions $\mathscr{N}_i$ must have the following property:

$$\lim_{\lambda\to 0} \mathscr{N}_i \to 0. \qquad (6)$$

Hence, a general variational approach to the physical phenomenon under consideration is established.

Remark. It should be noted that sometimes the variation δv_i does not vanish on some part of the boundary V. In this case the action integral (3) has to be adapted by the addition of new terms, in such a way that the Lagrangian equations (4) are valid. (For more details see [6], pp. 22–29.) In other words the stationary condition for the adapted action integral, with the first term equal to the right hand side of (3) yields the Euler–Lagrange equations (4).

To be more explicit let us consider the differential equation for an intensive nonstationary temperature field [15, 16]:

$$C\tau\frac{\partial^2 T}{\partial t^2} + C\frac{\partial T}{\partial t} = k\nabla^2 T, \qquad (7)$$

where T is the temperature, τ—relaxation time, C—heat capacity and k—heat conductivity.

Equation (7) may be derived from a variational principle, the Lagrangian of which is [29, 30, 32]

$$\mathscr{L} = \left[\frac{C\tau}{2}\left(\frac{\partial T}{\partial t}\right)^2 - \frac{k}{2}\sum_{j=1}^{3}\left(\frac{\partial T}{\partial x_j}\right)^2\right]e^{t/\tau}, \qquad (8)$$

where T plays the same role as v_i does in (2). Putting (8) into (4) we will get (7) in which case $\mathscr{N}\equiv 0$.

If we want to consider the classical case, the relaxation time plays the role of the parameter λ, since the former tends to zero. Thus the function $\mathscr{N}$ has the form

$$\mathscr{N} = -C\tau\frac{\partial^2 T}{\partial t^2}$$

the requirement (6) is fulfilled automatically, and we have the simple heat conduction equation

$$C\frac{\partial T}{\partial t} = k\nabla^2 T. \qquad (9)$$

In the case of variable thermal properties the classical heat conduction equation is of the form:

$$\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[k(T)\frac{\partial T}{\partial x_j}\right] = S(T)\frac{\partial T}{\partial t}, \qquad (10)$$

(k—thermal conductivity, S—volumetric heat capacity, product of density and heat capacity at constant pressure).

This equation may be derived from a Lagrangian of the form:

$$\mathscr{L} = \left\{\frac{k^2(T)}{2}\sum_{j=1}^{3}\left(\frac{\partial T}{\partial x_j}\right)^2 - \frac{\lambda}{2}S(T)\,k(T)\times\left(\frac{\partial T}{\partial t}\right)^2\right\}e^{t/\lambda}. \qquad (11)$$

Using (4) one gets

$$\sum_{j=1}^{3}\left\{\frac{\partial}{\partial x_j}\left[k(T)\frac{\partial T}{\partial x_j}\right]\right\} - S(T)\frac{\partial T}{\partial t} = \frac{\lambda}{k(T)}\left\{\frac{\partial}{\partial t}\left(Sk\frac{\partial T}{\partial t}\right) - \frac{1}{2}\frac{\partial}{\partial T}\times\left[S(T)k(T)\left(\frac{\partial T}{\partial t}\right)^2\right]\right\} \qquad (12)$$

hence when $\lambda\to 0$ we have (10).

3. NONLINEAR HEAT CONDUCTION PROBLEM, SEMI-INFINITE SOLID

As first example we shall study the heat conduction problem through semi-infinite bodies in one dimension with variable thermal properties. The governing differential equation is equation (10) for $j = 1$. A case of special interest is the semi-infinite slab at temperature T_0 whose face is suddenly raised to temperature T_1. In accordance with the well known notion of penetration depth ([28], p. 474) we now define a quantity $\delta(t)$ called the penetration distance. Its properties are such that for $x > \delta(t)$ the slab is at the equilibrium temperature and there is not heat transferred beyond this point. We will assume the case where the surface temperature is an arbitrary function of time

Hence, the boundary conditions of the problem are:

$$T(0,t) = T_1(t); \quad T[\delta(t), t] = T_0 = \text{const.} \tag{13}$$

In order to obtain an approximate solution we will assume a temperature profile in the form

$$T(x,t) = T_0 + \theta(t) \,.\, \mu^n, \tag{14}$$

where

$$\theta \equiv T_1(t) - T_0, \tag{15}$$

$$\mu = 1 - \frac{x}{\delta(t)}, \tag{16}$$

and n is an integer.

Let us assume that the thermal conductivity and volumetric heat capacity are of the form:

$$\frac{k(T)}{k_0} = \sum_{k=0}^{\infty} a_k \left(\frac{T - T_0}{\theta}\right)^k = \sum_{k=0}^{\infty} a_k \,.\, \mu^{nk}, \tag{17}$$

$$\frac{S(T)}{S_0} = \sum_{k=0}^{\infty} b_k \left(\frac{T - T_0}{\theta}\right)^k = \sum_{k=0}^{\infty} b_k \,.\, \mu^{nk}, \tag{18}$$

where k_0, S_0, a_k and b_k are given constants.

For this problem the action integral is

$$I = \int_{t_0}^{t_1} \int_0^{\delta(t)} \left\{ \frac{k^2(T)}{2} \left(\frac{\partial T}{\partial x}\right)^2 - \frac{\lambda}{2} S(T)\, k(T) \times \left(\frac{\partial T}{\partial t}\right)^2 \right\} e^{t/\lambda}\, dx\, dt, \tag{19}$$

where the time interval $t_0 \to t_1$ is chosen arbitrarily. In order to obtain an approximate solution we will apply the method of partial integration. Substituting (14), (17) and (18) into (19) and using (16) after integration with respect to x the equation (19) yields:

$$I = \int_{t_0}^{t_1} \left\{ \frac{k_0^2}{2} \frac{\theta^2}{\delta^2} A - \frac{\lambda S_0 k_0}{2} \left[\theta'^2 B + \frac{2\theta\theta' \,.\, \delta'}{\delta} C + \frac{\theta^2 \delta'^2}{\delta^2} D \right] \right\} e^{t/\lambda} \,.\, \delta\, dt \equiv \int_{t_0}^{t_1} \mathscr{L}(\delta, \delta', t)\, dt,$$

$$\times \left(\delta' \equiv \frac{d\delta}{dt}; \qquad \theta' \equiv \frac{d\theta}{dt} \right). \tag{20}$$

Where

$$A = n^2 \sum_{i,j=0}^{\infty} \frac{a_i \,.\, a_j}{n(i + j + 2) - 1}, \tag{21}$$

$$D = n^2 \sum_{i,j=0}^{\infty} b_i a_j \left[\frac{1}{n(i + j + 2) - 1} - \frac{2}{n(i + j + 2)} + \frac{1}{n(i + j + 2) + 1} \right], \tag{22}$$

$$C = n \sum_{i,j=0}^{\infty} b_i a_j \left[\frac{1}{n(i + j + 2)} - \frac{1}{n(i + j + 2) + 1} \right]. \tag{23}$$

The constant B has no influence on the results obtained below. In order to assure that the action integral (20) has a stationary value the Euler–Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \delta'} - \frac{\partial \mathscr{L}}{\partial \delta} = 0,$$

must be satisfied. Hence we have

$$\frac{S_0 D}{2} \theta^2(t) \frac{d\delta^2}{dt} + S_0 C \theta(t) \frac{d\theta}{dt}$$

$$\times\ \delta^2(t) - \frac{k_0 A}{2}\theta^2(t) = \lambda\{\ldots\ldots\}.$$

Putting $\lambda \to 0$ we have the ordinary differential equation

$$\frac{S_0 D}{2}\theta^2(t)\frac{d\delta^2}{dt} + S_0 C\theta(t)\frac{d\theta}{dt} \times \delta^2(t) - \frac{k_0 A}{2}\theta^2(t) = 0, \qquad (24)$$

the solution of which is for $\delta(0) = 0$:

$$\delta(t) = \left\{\frac{k_0}{S_0}\cdot\frac{A}{D}\theta(t)^{-2C/D}\int_0^t \theta(t)^{2C/D}\,.\,dt\right\}^{\frac{1}{2}}. \qquad (25)$$

Hence, the approximate solution is found. The solution (25) involves a very broad class of nonlinear problems which are defined for various forms of the functions $S(T)$ and $k(T)$ represented by the series (17) and (18).

The analysis of the result (25) will be given later.

4. ESTIMATION OF THE SOLUTIONS OBTAINED BY THE HELP OF THE VARIATIONAL PRINCIPLE

The estimate of the error incurred in the approximate solution of any problem is obviously very important. In this section we will follow the method suggested by Yang [33, 34] (see also [35]. p. 106), who demonstrated an improved integral method. Using Yang's analysis as a starting point we will discuss the problem of improving approximative profiles, the problem of choosing the optimum profile and minimization of the error integral. In performing this analysis we will show the utility of the variational principle presented in this paper.

(a) *Improving approximate profiles*

Following Yang we will transform the differential equation of heat conduction (10) using the transformation

$$\lambda_1 = \frac{x}{\delta(t)}; \qquad \tau_0 = t. \qquad (26)$$

The resulting equation is

$$P_1\frac{\partial^2 T}{\partial \lambda_1^2} + P_2\frac{\partial T}{\partial \lambda_1} = P_3, \qquad (27)$$

where

$$P_1 = k,$$

$$P_2 = \frac{\partial k}{\partial T}\cdot\frac{\partial T}{\partial \lambda_1} + S\lambda_1\delta\dot{\delta}, \quad \left(\dot{\delta} \equiv \frac{d\delta}{dt}\right) \qquad (28)$$

$$P_3 = S\frac{\partial T}{\partial \tau_0}\,.\,\delta^2.$$

The differential equation (27) may be used in obtaining the improved profile if we calculate P_1, P_2 and P_3 for the previously assumed profile. (For more details see [35], p. 106.)

We shall introduce the quantity

$$\varepsilon = P_1\frac{\partial^2 T}{\partial \lambda_1^2} + P_2\frac{\partial T}{\partial \lambda_1} - P_3 \qquad (29)$$

and the following error criterion in the form of the integral

$$J = \int_0^{\lambda_{1m}} \varepsilon^2\, d\lambda_1 \qquad (30)$$

where the upper bound λ_{1m} depends on the character of the chosen profile.

For the sake of simplicity we will study the improvement of a profile in the case of linear heat flow. Let us consider a semiinfinite slab initially at zero temperature whose face is suddenly raised to temperature T_s.

For this case putting

$$T_0 = 0; \quad k = k_0 = \text{const}, \theta(t) = T_s = \text{const};$$
$$S = S_0 = \text{const}, \quad \alpha = k_0/S_0, \qquad (31)$$

we obtain from (17) and (18)

$$a_0 = 1; \quad a_k = 0 \quad (k = 1, 2, 3, \ldots)$$
$$b_0 = 1; \quad b_k = 0 \quad (k = 1, 2, 3, \ldots). \qquad (32)$$

Choosing the cubic profile ($n = 3$)

$$T = T_s\left(1 - \frac{x}{\delta(t)}\right)^3 \qquad (33)$$

and using (21)–(23) and (25) we find that the penetration depth is

$$\delta = \sqrt{(21\alpha t)}. \tag{34}$$

Hence, from (28)

$$P_1 = 1,$$
$$P_2 = \frac{1}{\alpha}\lambda_1\delta\dot{\delta} \tag{35}$$
$$P_3 = 0.$$

Combining (34) and (35) the differential equation (27) becomes

$$\frac{d^2T}{d\lambda_1^2} + \frac{21}{2}\lambda_1\frac{dT}{d\lambda_1} = 0, \tag{36}$$

the solution of which for

$$\lambda_1 = 0, \quad T = T_s,$$
$$\lambda_1 = \infty, \quad T = 0,$$

is

$$T = T_s\left[1 - \operatorname{erf}\sqrt{\left(\frac{21}{4}\right)\lambda_1}\right]. \tag{37}$$

Substituting (34) into (37) obtain the exact solution to the problem:

$$T = T_s\left[1 - \operatorname{erf}\frac{x}{2\sqrt{(\alpha t)}}\right],$$

$$\left(\operatorname{erf}\eta = \frac{2}{\sqrt{(\pi)}}\cdot\int_0^\eta e^{-\alpha^2}\,d\alpha\right).$$

The efficiency of the method is obvious. Still, it should be noted that in more complex situations this procedure must be repeated several times, choosing improved profiles as the starting profile in each step.

(b) *Choosing the optimum profile*

The method developed in section (a) may be used for obtaining the optimal profiles. It is interesting to note that in recent years many methods have been developed for obtaining approximate solutions in irreversible physics. (Integral method, Biot's variational method, the method of weighting residuals, collocations method, method of moments, etc.) Applying any of the methods cited an important question arises. How to choose the best order of the polynomial used to represent the profile. Concerning this question, several authors thought that increasing the order of polynomial would lead to more accurate solutions. This statement is not valid according to Goddman [35] who has shown that there is no *a priori* guarantee that increasing the order of the polynomial will improve the accuracy.

In this section we will show that Goodman's conclusion is correct and we will discuss a method of finding the best degree of a polynomial for a specific problem. We will focus our attention on two nonlinear problems.

(i) Yang's case [18]

$$k = k_0\left(1 + \alpha\frac{T}{T_1}\right); \quad S = S_0 = \rho C_0 = \text{const},$$
$$T_0 = 0; \quad \theta = T_1 = \text{const}. \tag{38}$$

In this case we have that:

$$b_0 = 1; \quad b_i = 0; \quad (i = 1, 2, 3, \ldots)$$
$$a_0 = 1; \quad a_1 = \alpha; \quad a_i = 0; \quad (i = 2, 3, 4, \ldots) \tag{39}$$

and the solution (25) is of the form:

$$\delta(t) = \left(\frac{k_0}{S_0}\right)^{\frac{1}{2}}\cdot\sqrt{(At)} \tag{40}$$

where:

$$A = \frac{\dfrac{1}{2n-1} + \dfrac{2.\alpha}{3n-1} + \dfrac{\alpha^2}{4n-1}}{\dfrac{1}{2n-1} - \dfrac{1}{n} + \dfrac{1}{2n+1} + \alpha\left(\dfrac{1}{3n-1}\;\;\dfrac{2}{3n} + \dfrac{1}{3n+1}\right)}. \tag{41}$$

The problem we wish to solve is to find the power n in such a way that the error is minimized. Using (16), (14), (40), (38), (26) and substituting in (28) and (29) we find the quantity ε. Performing the integration of (30) from zero to $\lambda_{1m} = 1$ we obtain the error criterion in the form:

$$J = k_0^2 T_1^2 n^2 \left\{ (n-1)^2 \left(\frac{1}{2n-3} + \frac{2\alpha}{3n-2} + \frac{\alpha^2}{4n-3} \right) + 2(n-1) \left[\alpha n \left(\frac{1}{3n-3} + \frac{\alpha}{4n-3} \right) + \frac{A}{2} \left(\frac{1}{2n-1} - \frac{1}{2n-2} + \frac{\alpha}{3n-1} - \frac{\alpha}{3n-2} \right) \right] + \alpha n A \left(\frac{1}{3n-1} - \frac{1}{3n-2} \right) + \frac{A^2}{4} \left(\frac{1}{2n+1} - \frac{1}{n} + \frac{1}{2n-1} \right) + \frac{\alpha^2 n^2}{4n-3} \right\}.$$

The minimum of the function J with respect to n for various α is given in the following table:

α	$-0{\cdot}5$	0	$0{\cdot}5$	1	$1{\cdot}5$
n	5·2	2·02	1·76	1·67	1·63
$\frac{J_{\min}}{k_0^2 T_1^2}$	0·15783	0·66569	2·105717	4·35087	7·435511

Using (38)–(41) a comparison with the exact solution [18] is given for various value of n. From the Figs. 1–3 which are plotted for $\alpha = \pm 0{\cdot}5$, and 0 it is seen that for the value of n for which the function J has the minimum, the agreement with the exact solution is quite satisfactory. To the authors knowledge the solution for $\alpha = 1$ and $\alpha = 1{\cdot}5$ has not been found. The Fig. 4 shows the temperature distributions for these cases. Since the function J has a minimum value it should be expected that this solution is satisfactory.

(ii) Biot's case.

$$T_0 = 0; \quad \theta = T_1 = \text{const}; \quad k = k_0 = \text{const}, \tag{42}$$

$$S = S_0\left(1 + \frac{T}{T_1}\right); \quad \rho = \text{const}; \quad S_0 = \rho C_0.$$

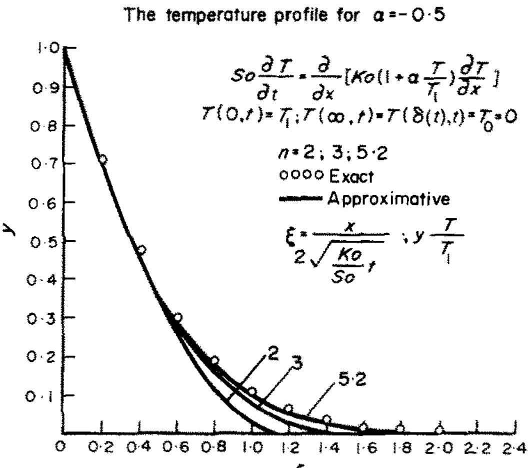

FIG. 1. The temperature profile for $\alpha = -0{\cdot}5$.

In this case we have that

$$a_0 = 1; \qquad a_k = 0, \quad (k = 1, 2, 3, \ldots)$$

$$b_0 = b_1 = 1; \quad b_k = 0, \quad (k = 2, 3, 4, \ldots) \tag{43}$$

and the solution (25) is

$$\delta(t) = \left(\frac{k_0}{\rho C_0}\right)^{\frac{1}{2}} \cdot \sqrt{(At)} \tag{44}$$

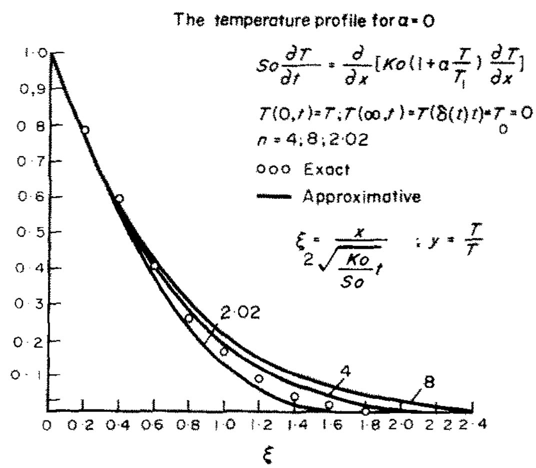

FIG. 2. The temperature profile for $\alpha = 0$.

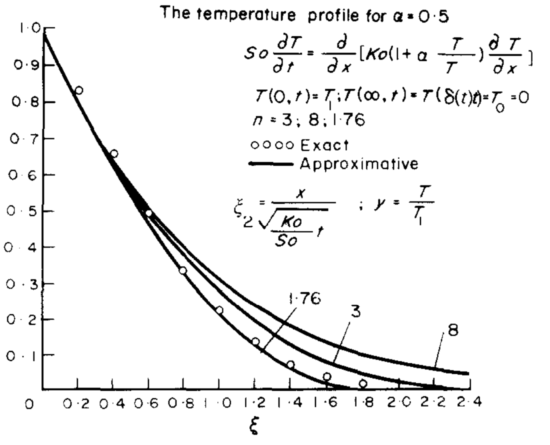

FIG. 3. The temperature profile for $\alpha = 0{\cdot}5$.

where

$$A = \frac{\dfrac{1}{2n-1}}{\dfrac{1}{2n-1} - \dfrac{1}{n} + \dfrac{1}{2n+1} + \dfrac{1}{3n-1} - \dfrac{2}{3n} + \dfrac{1}{3n+1}}. \qquad (45)$$

Hence, as in the previous problem we will find the parameter n in such a way as to minimize the error.

Using (14)–(16), (26), (44), (45) and (42) and substituting into (28) and (29) we find the corresponding quantity for ε. Performing integration in (30) from zero to $\lambda_{1m} = 1$ the error criterion is

$$J = k_0^2 T_1^2 n^2 \left\{ \frac{(n-1)^2}{2n-3} - (n-1) \times A \left(\frac{1}{2n-2} - \frac{1}{2n-1} + \frac{1}{3n-2} - \frac{1}{3n-1} \right) + \frac{A^2}{4} \left(\frac{1}{2n+1} + \frac{1}{2n-1} + \frac{2}{3n+1} + \frac{2}{3n-1} + \frac{1}{4n+1} + \frac{1}{4n-1} - \frac{17}{6n} \right) \right\}. \qquad (46)$$

The minimum of function J is achieved for

$$n = 2{,}1$$

and for this value of n, the value of J is

$$J = k_0^2 T_1^2 . 0{\cdot}63988,$$

and from (44) and (45), for $n = 2{\cdot}1$ we have

$$\delta = 2{\cdot}913 \left(\frac{k_0}{\rho C_0} t \right)^{\frac{1}{2}}.$$

The same problem was solved by Biot using his variational technique. For $n = 2$ Biot has found [10]

$$\delta = 2{\cdot}97 \left(\frac{k_0}{\rho C_0} t \right)^{\frac{1}{2}}.$$

For a profile of the same type, using the variational principle presented in this paper (see [29]) the result is:

$$\delta = 2{\cdot}80 \left(\frac{k_0}{\rho C_0} t \right)^{\frac{1}{2}}.$$

Unfortunately this problem is not liable to an exact solution and direct comparison is impossible. But, by minimizing the error criterion it seems logical to suppose that the given result is correct.

(c) *Comparisons with other methods*

It would be interesting to make comparisons between the results obtained by the help of variational principle and other methods.

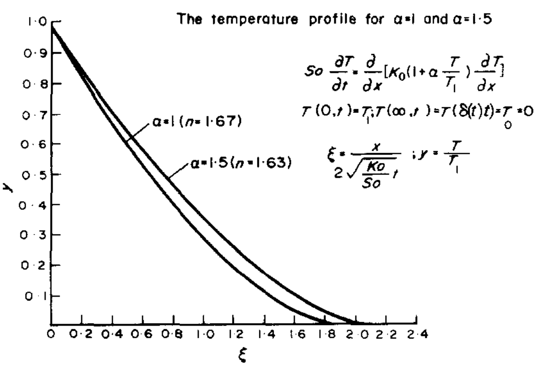

FIG. 4. The temperature profile for $\alpha = 1$ and $\alpha = 1{\cdot}5$.

For a given temperature profile i.e. for fixed n, the various methods give qualitatively same results, but the numerical values are often different. One can reasonably ask to find the "best" approximate solution for a given profile. We will show that the solutions obtained by the help of variational principle are the "best" in the sense of Yang's analysis.

(i) Let us study the problem in section (a), for which we found the solution (34) in the form

$$\delta = \sqrt{(21\alpha t)}. \qquad (47)$$

Solving the same problem [and using the same profile (33)], other authors have found

$$\delta = \sqrt{(A\alpha t)}. \qquad (48)$$

For example, using the integral method Goodman found [35], p. 59

$$A = 24.$$

The problem we wish to solve is to find the constant A for which the error is a minimum.

For the fixed profile (33) and for δ defined by (48) and using (35), the error criterion will be

$$J = T_s^2\left(12 - \frac{9}{10}A + \frac{3}{140}A^2\right), \qquad (49)$$

where the integration in (30) has been performed from 0 to 1. The function J has a minimum value for

$$A = 21.$$

Hence, we have shown that the result $A = 21$ obtained by the help of the variational principle makes a minimum of the error integral (49).

For the quadratic profile

$$T = T_s\left(1 - \frac{x}{\delta(t)}\right)^2$$

the variational technique gives

$$\delta = \sqrt{(10\alpha t)}.$$

For the same problem, (and for the same profile), Biot found

$$\delta = \sqrt{\left(\frac{147}{13}\alpha t\right)}.$$

Choosing the expression for penetration distance in the form of (48), it is easy to show that the error criterion will give a minimum for

$$A = 10$$

which is the same value obtained by the help of variational principle.

(ii) For the case of nonlinear problem studied in (ii) of section (b) we found that for the quadratic profile $n = 2$, the solution is

$$\delta = 2{\cdot}8\left(\frac{k_0}{\rho C_0}t\right)^{\frac{1}{2}} \qquad (50)$$

Using the same profile Biot found

$$\delta = 2{\cdot}97\left(\frac{k_0}{\rho C_0}t\right)^{\frac{1}{2}}.$$

Supposing that the penetration distance has the form

$$\delta = \left(A\frac{k_0}{\rho C_0}t\right)^{\frac{1}{2}} \qquad (A = \text{const.})$$

and forming the error criterion, we have

$$J = 4 + 4A(-\tfrac{13}{60}) + A^2(\tfrac{28}{5} - \tfrac{10}{3} + \tfrac{17}{7} + \tfrac{1}{9} - \tfrac{19}{4}).$$

This function has a minimum value for

$$A^{\frac{1}{2}} = 2{\cdot}8.$$

Hence, in all three examples it is seen that the results obtained by the help of the variational principle, are optimal for a given temperature profile.

5. FRIEDMANN'S NONLINEAR PROBLEM

The solution (25) of equation (10) covers a very broad class of problems. For example the solutions given in [20–25] are covered by the solution cited in this paper. For example it is possible to easily obtain the solution of the strongly nonlinear problem considered by Friedmann in [19].

This problem is defined by

$$S(T) = C_0\rho_0\beta . mT^{m-1}, \quad C_0, \rho_0, \beta = \text{const},$$
$$k(T) = \lambda_0 p\alpha T^{p-1}, \qquad \lambda_0, \alpha = \text{const}. \qquad (51)$$

$T_0 = 0$, $\theta = T_1 = \text{const}$, $n = q/p$, $m = \text{const}$, $p = \text{const}$. We will assume the temperature profile in the form:

$$\left(\frac{T}{T_1}\right)^p = \mu^q \qquad (q = \text{const}). \tag{52}$$

Using the last three expressions (51) it is obvious that the profile (52) is of the same type as (14). Hence the general solution (25) is valid for this problem. For this problem the coefficients a_k, b_k, S_0 and k_0 have the form:

$$
\begin{aligned}
S_0 &= C_0 \rho_0; \qquad k_0 = \lambda_0 \\
a_k &= \begin{cases} 0 & \text{for} \quad k \neq p - 1 \\ \alpha p T_1^{p-1} & \text{for} \quad k = p - 1 \end{cases} \\
b_k &= \begin{cases} 0 & \text{for} \quad k \neq m - 1 \\ \beta m T_1^{m-1} & \text{for} \quad k \neq m - 1. \end{cases}
\end{aligned}
\tag{53}
$$

By means of (21)–(23), (51), (53) the solution (25) becomes

$$\delta(t) = \left(\frac{\lambda_0}{\rho_0 C_0} \frac{\alpha}{\beta} \frac{p}{m} T_1^{p-m} \times \frac{\dfrac{t}{2q-1}}{\dfrac{1}{\dfrac{mq}{p} + q - 1} - \dfrac{2}{\dfrac{mq}{p} + q} + \dfrac{1}{\dfrac{mq}{p} + q + 1}} \right)^{\frac{1}{2}}. \tag{54}$$

The temperature field corresponding to this problem is plotted in Figs. 5 and 6 for $q = 2$, and

$$\xi = \frac{x}{\left(\dfrac{\lambda_0}{\rho_0 C_0} \dfrac{\alpha}{\beta} \dfrac{p}{m} T_1^{p-m} . t\right)^{\frac{1}{2}}},$$

the comparison with the exact solution for aluminium ($p = 0{\cdot}905$, $m = 1{\cdot}03$) given in [19] is seen to be quite satisfactory, i.e. it is impossible to detect any error when this temperature profile is superimposed on the graph which Friedmann presents. Unfortunately for steel ($p = 0{\cdot}84$, $m = 1{\cdot}125$) there is no known solution for comparison.

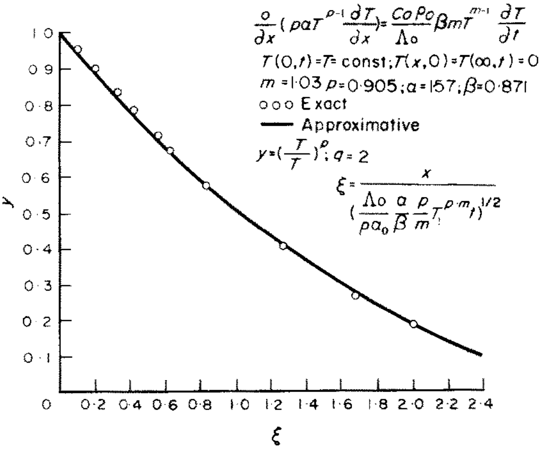

FIG. 5. The temperature distribution for differential equation of Friedmann's type

$$\frac{\partial}{\partial t}(\beta T^m) = \frac{\lambda_0}{\rho_0 C_0} \cdot \frac{\partial^2}{\partial x^2}(\alpha T^p)$$

for aluminium $p = 0{\cdot}905$, $m = 1{\cdot}03$.

6. THE SURFACE RADIATION

As an interesting application of variational principle defined in previous chapters, we are going to discuss the problem of surface radiation. This problem is defined by the equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}; \qquad \frac{K}{S} = \alpha = \text{const}, \tag{55}$$

with the nonlinear boundary conditions

$$k \frac{\partial T}{\partial x} = h(T^m - T_0^m) \qquad \text{for} \qquad x = 0, \tag{56}$$

and

$$T[\delta(t), t] = \theta_0 = \text{const}, \tag{57}$$

where $k = \text{const}$ is the thermal conductivity and $\alpha = k/S = \text{const}$, where S is volumetric heat capacity and h is a constant factor of proportionality. T is the absolute temperature of the body and T_0 is the absolute temperature of surrounding medium. First of all we shall introduce a Langrangian of the form (11)

$$\mathscr{L} = \left[\frac{\lambda}{2}\left(\frac{\partial T}{\partial t}\right)^2 - \frac{\alpha}{2}\left(\frac{\partial T}{\partial x}\right)^2\right] e^{t/\lambda},$$

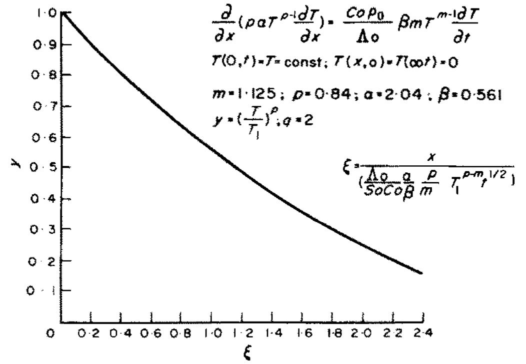

FIG. 6. The temperature distribution for differential equation of Friedmann's type

$$\frac{\partial}{\partial t}(\beta T^m) = \frac{\lambda_0}{\rho_0 C_0}\cdot\frac{\partial^2}{\partial x^2}(\alpha T^p)$$

for steel $p = 0{\cdot}84$, $m = 1{\cdot}125$.

the action integral of which is

$$I = \int_{t_0}^{t_1}\int_0^{\delta} \mathscr{L}\,\mathrm{d}x\,\mathrm{d}t. \tag{58}$$

In the sense of classical mechanics the boundary condition (56) plays the role of a nonholonomic constraint i.e. if we introduce the value (56) for $\partial T/\partial x$ into the Langrangian of the problem the Langrangian will lose its utility because the "transformation" (56) is nonintegrable (nonholonomic). The general discussion about the impossibility of describing the physical process by Langrange's equation in the presence of non-holonomic "differential transformations" like (56) can be found in [26] and [27]. In order to overcome this difficulty we shall follow the method suggested by Raphalsky and Zyszkowsky [12].

Following the procedure in [12] two generalized coordinates are introduced: the penetration distance δ and the absolute surface temperature q. We will seek the approximative solution in the form

$$T = \theta_0 - (\theta_0 - q)\left(1 - \frac{x}{\delta(t)}\right)^n, \tag{59}$$

where n is an integer. Introducing (59) into (58) and integrating with respect to x one gets

$$\begin{aligned} I = \int_{t_0}^{t_1} \Bigg\{ &\frac{\lambda}{2}\Bigg[\dot{q}^2\frac{\delta}{2n+1} - 2\dot{\delta}\dot{q}(\theta_0 - q) \\ &\times\left(\frac{1}{2} - \frac{n}{2n+1}\right) + \frac{(\theta_0 - q)^2}{\delta}.\dot{\delta}^2 \\ &\times\left(\frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n+1}\right)n^2\Bigg] \\ &- \frac{\alpha n^2}{2(2n-1)}\frac{(\theta_0 - q)^2}{\delta}\Bigg\}\,\mathrm{e}^{t/\lambda}\,\mathrm{d}t \\ \equiv \int_{t_0}^{t_1} &\mathscr{L}(\delta, \dot{\delta}, t)\,\mathrm{d}t. \quad \left(\dot{\delta} \equiv \frac{\mathrm{d}\delta}{\mathrm{d}t};\ \dot{q} \equiv \frac{\mathrm{d}q}{\mathrm{d}t}\right). \end{aligned} \tag{60}$$

Applying the Euler–Langrange equation with respect to coordinate δ

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\dot{\delta}} - \frac{\partial\mathscr{L}}{\partial\delta} = 0$$

after the limiting process $\lambda \to 0$ we have the differential equation

$$\begin{aligned} -2n\left(\frac{1}{2n} - \frac{1}{2n+1}\right)\dot{q} &+ 2n^2 \\ \times\left(\frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n+1}\right)&\frac{\dot{\delta}}{\delta}(\theta_0 - q) \\ &- \frac{\alpha n^2}{2n-1}\frac{\theta_0 - q}{\delta^2} = 0. \end{aligned} \tag{61}$$

The existance of the nonholonomic constraint (56) leads, substituting the temperature profile (59) into (56), to the algebraic relation, [12], equation (28).

$$\delta = \frac{nk}{h}\frac{\theta_0 - q}{q^m - T_0^m}. \tag{62}$$

Introducing (62) into (61) and setting

$$z = \frac{q}{\theta_0};\quad z_0 = \frac{T_0}{\theta_0};\quad \tau = \frac{h^2\alpha}{k^2}\,\theta_0^{2(m-1)}.t, \tag{63}$$

the resulting differential equation yields

$$\left\{(1 - z)\left[2n^2\left(\frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n+1}\right)\right.\right.$$
$$\times (mz^m - z^m - mz^{m-1} + z_0^m) - (z^m - z_0^m)$$
$$\left.\left.\times 2n\left(\frac{1}{2n} - \frac{1}{2n+1}\right)\right] dz\right\}$$
$$\times \frac{1}{(z^m - z_0^m)^3} = \frac{d\tau}{2n-1}. \tag{64}$$

This problem was selected to demonstrate the variational technique in the presence of nonlinear boundary conditions. Hence, it is not within the scope of this paper to analyze the solution of (64) in details. We will only compare the asymptotic solutions for short and long times.

For $\tau \to 0$ the solution, when $T_0 = 0$, is

$$z = 1\text{–}1{\cdot}190\tau^{\frac{1}{2}} \quad \text{for} \quad n = 2,$$
$$z = 1\text{–}1{\cdot}129\tau^{\frac{1}{2}} \quad \text{for} \quad n = 3.$$

The exact short time solution is [36]

$$z = 1\text{–}1{\cdot}128\tau^{\frac{1}{2}},$$

and does not depend upon the power m. The long time solution, $\tau \to \infty$, of (64) for $T_0 = 0$ is:

$$z = (1{\cdot}58\tau^{\frac{1}{2}})^{-1/m} \quad \text{for} \quad n = 2$$
$$z = (1{\cdot}53\tau^{\frac{1}{2}})^{-1/m} \quad \text{for} \quad n = 3.$$

The exact long time solution is [36]

$$z = (1{\cdot}77\tau^{\frac{1}{2}})^{-1/m}.$$

It is seen that the results obtained here are in good agreement with the exact solution.

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REFERENCES

1. B. A. Finlayson and L. E. Scriven, On the search for variational principles, *Int. J. Heat Mass Transfer* **10**, 799–821 (1967).
2. P. Rosen, Use of restricted variational principles for the solution differential equations, *J. Appl. Phys.* **25**, 336–338 (1954).
3. L. G. Shambers, A variational principle for the conduction of heat, *Q. J. Mech. Appl. Math. IX*, Pt. 2, 234–235 (1956).
4. P. Glansdorff and I. Prigogine, On a general evolution criterion in macroscopic physics, *Physica* **30**, 351 (1964).
5. P. Glansdorff and I. Prigogine, Variational properties and fluctuation theory, *Physica* **31**, 1242 (1965).
6. R. S. Schechter, *The Variational Method in Engineering*. McGraw-Hill, New York (1967).
7. I. Gyarmati, *Non-equilibrium thermodynamics*. Springer-Verlag, Berlin; Heidelberg; New York (1970).
8. H. L. Dryden, F. D. Murnaghan and H. Bateman, *Hydrodynamics*, Part II, pp. 165–168. Dover (1956).
9. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, part I, p. 298. McGraw-Hill, New York (1953).
10. M. A. Biot, New methods in heat flow analysis with application to flight structures, *J. Aeronaut. Sci.* **24**, 857–873 (1957).
11. T. J. Lardner, Biot's variational principle in heat conduction, *AIAA Jl* **1**, 196–206 (1963).
12. P. Rafalsky and W. Zyszkowsky, Langrangian approach to the nonlinear boundary heat-transfer problem, *AIAA Jl* **6**, (8), 1606–1608 (1968).
13. S. D. Nigam and H. C. Agrawal, A variational principle for conduction of heat, Parts I and II, *J. Math. Mech.* **9** (6); 869–883 (1960).
14. R. L. Seliger and F. R. S. Whitham, Variational principles in continuum mechanics, *Proc. R. Soc.* **305A**, 1–25 (1968).
15. M. Chester, Second sound in solids, *Phys. Rev.* **131**, 2013–2015 (1963).
16. M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Arch. Rat. Mech. Anal.* **31**, 113–126 (1968).
17. R. J. Donnelly, R. Herman and I. Prigogine, *Non-Equilibrium Thermodynamics Variational Techniques and Stability*. Univ. of Chicago Press (1967).
18. K. T. Yang, Transient conduction in a semi-infinite solid with variable thermal conductivity, *J. Appl. Mech.* **25**, 146 (1958).
19. N. E. Friedmann, Quasilinear heat flow, *Trans. Am. Soc. Mech. Engrs* **80**, 635–645 (1958).
20. J. S. Aronofsky and R. Jenkins, Unsteady flow of gas through porous media, one-dimensional case, Proc. First U.S. National Congress for Applied Mech. pp. 763–771. J. W. Edwards Bros., Ann Arbor, Mich. (1952).
21. G. I. Barenblatt, On self-similar motions of a compressible fluid in porous medium, *Akad. Nauk USSR, Prikl. Math. i Mehk.* **16**, 679–698 (1952).
22. N. E. Friedmann, Quasilinear heat flow, Doctoral dissertation, Univ. of Cal, Los Angeles, Cal. (November 1956).
23. B. Ya. Lyubov, Computation of the rate of solidification of an ingot taking into account the temperature dependance of the thermophysical parameters of the metal,
24. P. Y. Polubarinova-Kochina, On a nonlinear partial

differential equation arising in the theory of filtration, *Dokl. Akad. Nauk USSR* (N.S.) **63**, 623–626 (1948).
25. M. R. HOPKINS, Heat conduction in a medium having thermal properties depending on the temperature, *Proc. Phys. Soc., Lond.* **50**, 703 (1938).
26. P. APPELL, *Traité de Mécanique Rationelle*, Tome Deuxiéme. Gautier-Villars, Paris (1953).
27. P. APPELL, Sur une forme générale des équations de la dynamique et sur le principle de gauss, *J. Reine Angew. Math.* **122**, 205–208 (1900).
28. V. S. ARPACI, *Conduction Heat Transfer*. Addison-Wesley (1966).
29. B. VUJANOVIC, An approach to linear and nonlinear heat transfer problem using a lagrangian, *AIAA Jl* **9**, 131–134 (1971).
30. transfer via a variational principle, *AIAA Jl* **9**, 327–330 (1971).
31. Dj. DJUKIC and B. VUJANOVIC, On a new variational principle of Hamiltonian type for classical field theory, *Z. Angew. Math. Mech.* **51**, 611–616 (1971).
32. B. VUJANOVIC and Dj. DJUKIC, A variational principle for the theory of laminar boundary layers in incompressible fluids, *Publs. Inst. Math.* **11**, (25) 73–83 (1971).
33. K. T. YANG, *J. Appl. Mech.* **28**, 9 (1961).
34. K. T. YANG, *J. Heat Transfer* **84**, 353 (1962).
35. R. T. GOODMAN, Application of integral methods to transient nonlinear heat transfer, *Advances in Heat Transfer*, edited by T. IRVINE and J. HARTNETT, p. 60. Academic Press, New York, London (1964).
36. H. S. CARSLAW and J. C. JAEGER, *Conduction of Heat in Solids*, 2nd. Ed. Oxford University Press (1959).

SUR UN PRINCIPE VARIATIONNEL DE TYPE HAMILTONIEN POUR UNE PROBLÈME NON LINÉAIRE DE TRANSFERT THERMIQUE

Résumé—Dans cet article on établit un nouveau problème de Lagrange de conduction thermique non linéaire. A l'aide du concept de profondeur de pénétration on donne une procédure de calcul pour résoudre l'équation non linéaire de la chaleur. On discute aussi un problème de conditions aux limites non linéaires (rayonnement de surface). Par application de la méthode Yang (33), (34) et (35) on montre que les solutions peuvent être améliorées. On discute aussi le choix du meilleur polynôme pour la description de la distribution de température.

A la lumière de la théorie de Yang, les solutions obtenues au moyen du principe variationnel ont quelque degré d'optimalité en comparaison avec d'autres solutions approchées.

ÜBER EIN VARIATIONSPRINZIP VOM HAMILTONSCHEN TYP FÜR NICHTLINEARE WÄRMEÜBERTRAGUNGSPROBLEME

Zusammenfassung—In dieser Arbeit wird ein neuer Lagrange-Operator für nichtlineare Wärmeleitungsprobleme aufgestellt. Unter Verwendung des Prinzips der Eindringtiefe wird ein Berechnungsschema zur Lösung der nichtlinearen Wärmeleitungsgleichung mitgeteilt. Ausserdem wird ein Problem mit nichtlinearen Randbedingungen (Strahlung) diskutiert.

Unter Verwendung der Methode von Yang [33], [34] und [35] wird gezeigt, dass die Lösungen verbessert werden können. Es wird ferner eine Methode erörtert, wie man das beste Approximationspolynom zur Beschreibung der Temperaturverteilung zu wählen hat.

Aus der Sicht der Yangschen Theorie besitzen die mit Hilfe des Variationsprinzips gewonnenen Lösungen eine gewisse Optimalqualität im Vergleich mit anderen Näherungslösungen.

О ВАРИАЦИОННОМ ПРИНЦИПЕ ГАМИЛЬТОНОВСКОГО ТИПА ДЛЯ НЕЛИНЕЙНОЙ ЗАДАЧИ ПЕРЕНОСА ТЕПЛА

Аннотация—В работе получена новая функция Лагранжа для нелинейной задачи теплопроводности. Используя понятие глубины проникновения, разработана методика расчёта для решения нелинейного уравнения переноса тепла. Рассмотрена также задача с нелинейными граничными (излучение поверхности). Показано, что решения можно улучшить, используя метод Янга [33–35]. Рассмотрен также метод выбора наилучшего полинома для описания распределения температуры.

Приближенные решения, полученные с помощью вариационного принципа с использованием метода Янга, являются оптимальными.